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Mathematical modeling of lactation curves of dairy animals

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Abstract

An overview of mathematical models on cattle lactation is herein provided. A technique for constructing lactation curves using logical assumption and Verhulst's logistic function for predicting the cows' milk production is described. A new modeling lactation method that is based on solving the equation that describes the rate of change of milk production, depending on time elapsed since calving is presented. The statistical parameters of the adequacy of the models recommended in the literature for describing the lactation curve of cows and our proposed models were calculated Approximation accuracy allows for the identification of models that most reliably describe the lactation curve in the example of cattle.

Key words: production models, lactation curves, dairy cattle breeding, adequacy of models.

Математическое моделирование кривых лактации молочных животных.

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Аннотация:

Здесь представлен обзор математических моделей лактации крупного рогатого скота. Описан метод построения кривых лактации с использованием логических предположений и логистической функции Ферхюльста для прогнозирования надоев коров. Представлен новый метод моделирования лактации, основанный на решении уравнения, описывающего скорость изменения надоев молока в зависимости от времени, прошедшего с момента отела. Рассчитаны статистические параметры адекватности рекомендованных в литературе моделей для описания кривой лактации коров и предложенных нами моделей. Точность аппроксимации позволяет идентифицировать модели, наиболее достоверно описывающие кривую лактации на примере крупного рогатого скота.

Ключевые слова: Производственные модели, кривые лактации, молочное скотоводство, адекватность моделей.

Introduction

The breeding of dairy cattle is an important component of agricultural production having a significant impact on its economic efficiency. The potential milk yield of a cow during the lactation period continuously changes according to the change in its physiological state. The regularity of these changes is reflected in the lactation curve.

The lactation curve for a dairy cow increases rapidly from calving until its peak. Then follows a gradual decline up until the cow's milk stasis in ten months after calving with a dry period of 45 days. Changes in daily milk yield are determined by changes in the number and activity of the cells of the mammary glands. Attempts at a mathematical description of this curve often endeavor to predict milk yields, dietary requirements and cash flow. As a rule, gaining a better understanding of the lactation process at the quantitative level while so doing is not pursued as a goal [15].

Most of the works devoted to the lactation curve modeling in full feeding conditions are empirical or semi-empirical. In this paper, we present both existing and proposed lactation models.

Many authors have devoted their works to the given issue. To date, more than twenty formulas have been offered. All models can be conventionally divided into three groups depending on the method of their derivation: *empirical* (models that have been obtained after statistical processing of results from a large amount of experimental data; *logical* (models based on intuitive ideas fundamental to lactation curve tracing); *semi-empirical* (based on the principles and methods of the theory of population growth) [28].

1. Empirical Models.

Historical Development of Empirical Models for Lactation Process

In all the models described it is assumed that $\tilde{y}$ denotes daily milk yield ***t***, denotes time in days after parturition, and $a, b, c, d, k, \ldots$, and a_i, b_i denote model parameters.

The first attempt to develop a mathematical model to describe the lactation curve was launched by Brody, Ragsdale and Turner in 1923. They used the following model for this purpose:

M1 $$y = ae^{-ct} \qquad (1.1)$$

This model resulted in a good attempt to describe the declining phase of lactation, it was unable to model the initial rise in production to peak yield. Wood attributes this model to Gaines, but work by Gaines in this field was only published in 1927, whereas Brody et al. already published their paper in 1923. To overcome this limitation, Brody, Turner and Ragsdale presented an improved version of their model in 1924. This time the model made provision for the initial rise to peak production by incorporating an inclining function into the model:

M2 $$y = ae^{-bt} - ae^{-ct} = a\left(e^{-bt} - e^{-ct}\right) \quad (1.2)$$

Although this was a great improvement on their first model, later researchers such as Cobby and Le Du (1978):

M3 $$y = a\left(1 - e^{-bt}\right) - c = \left(a - c\right) - ae^{-bt} \quad (1.3)$$

found on fitting this model to lactation data of cows, that it resulted in underestimation of milk yield in mid-lactation and overestimated milk yield in late lactation.

This was followed by a relatively complex model than (1.1) introduced by Sikka (1950):

M4 $$y = a\,e^{bt - ct^2} \quad (1.4)$$

In 1958 Fisher proposed to improve the model (1.3) in the following form, by substituting the exponential decline built into this model with a linear decline:

M5 $$y = a - bt - ae^{-ct} = a\left(1 - e^{-ct}\right) - bt \quad (1.5)$$

The first attempt to formalize the lactation curve appears to belong solely to Vučić and Bačić (1961) who developed Gaines' model into the following form:

M6 $$y = a\,t\,e^{-ct} \quad (1.6)$$

This model seems to be the first attempt to develop a model that varies both directly and exponentially with time. Unfortunately, this equation does not exactly conform to the actual data [15].

In an effort to improve on all models that existed at the time, in the 1960s it was proposed an empirical model described by rational functions in the following form:

M7 $$y = \frac{a_0 + a_1 t + a_2 t^2 + \ldots a_m t^m}{b_0 + b_1 t + b_2 t^2 + \ldots b_n t^n} \quad (1.7)$$

Using model (1.7), we can consider various special cases depending on changes in values of coefficients (ai, bi) and degrees (*m, n*).

For example, when for the coefficients: $a_0, a_2, \ldots, a_m = 0,\ a_1 = 1,$ and $b_0 = a,\ b_1, = b,\ b_2 = c,\ b_3, \ldots, b_n = 0,$ for the degrees: *m=1* and *n=2*, then from equality (1.7) follows the inverse polynomial model proposed by Nelder (1966):

M8 $$y = \frac{t}{a + bt + ct^2} \quad (1.8)$$

For other combinations of values m and n from equation (1.7) can be easily obtained empirical model by Bianchini (1984):

M9 $$y = a + bt + \frac{c}{t} \tag{1.9}$$

and a number of other models (Narushin and Takma, 2003).

The main impetus to the development of empirical modeling of lactation did Wood (1967), who suggested using the gamma function:

M10 $$y = a t^b e^{-ct} \tag{1.10}$$

In the 1970s in search for an improvement on the existing at that time lactation models continued with the proposal of the polynomial model of the following form:

M11 $$y = a_0 + a_1 t + a_2 t^2 + ... a_n t^n = \sum_{i=0}^{n} a_i t^i \tag{1.11}$$

For example, for n=2 models by Dave (1971), for n=3 models by Dag et al. (2005).

Numerous studies [4, 12, 17, 20, 22, 25-26, 29 et al.] suggest linear and nonlinear empirical models, which contain different combination of algebraic and transcendental functions for the approximation of lactation data.

These include the most common models by Wilmink (1978), Ali and Scheaffer (1987), Guo and Swalve (1995), Mikayilov (2013) and (Memmedova and Mikayilov, 2014), which respectively have the form:

M12 $$y = a + ct + be^{-0,61t} \tag{1.12}$$

M13 $$y = a_0 + a_1 t + a_2 t^2 + b_1 \ln t + b_2 (\ln t)^2 \tag{1.13}$$

M14 $$y = a + b\sqrt{t} + c \ln t \tag{1.14}$$

M15 $$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + b_1 \ln t + b_2 (\ln t)^2 + b_3 (\ln t)^3 \tag{1.15}$$

2. Logical models

Based on the synthesis of numerous empirical models of lactation curves obtained under various conditions by scientific and production institutions, the logical premises for lactation curve tracing have been identified which means its analytical form can be defined.

Let us denote the amount of milk obtained from a cow from calving till ***t*** moment, ***kg/day***, where ***t*** is hours per day, by $y = f(t)$. If the time is increased by Δ***t***, the product amount will be equal to $f(t + \Delta t)$, and the gain for the product amount due to increasing the time by Δ***t*** will be: $\Delta y = f(t + \Delta t) - f(t)$.

As is known, the value y/t – is the average production capacity per time $[0; t]$. In this case, it is reasonable to assume that the value of ***y*** is proportional to the product of the average production capacity by the value Δ***t***, i.e. the following equality may be derived:

$$\Delta y = \alpha \cdot \frac{y}{t} \cdot \Delta t \tag{2.1}$$

i.e. the milk gain due to the extra increased time is directly proportional to the product of its average production capacity by the value of extra Δ***t*** time. Here ***α*** is a proportionality ratio. This means that when increasing the time, the product output increases but at the same time the Δ***y*** gain is less than its "natural" rate ***y*Δ*t/t.***

When being continuously represented (2.1), its form i**s** as follows:

$$\Delta y = \alpha(t) \cdot \frac{y}{t} \cdot \Delta t \quad \text{or} \quad \frac{\Delta y}{\Delta t} = \alpha(t) \cdot \frac{y}{t} \tag{2.2}$$

Dividing the equality $\Delta y = f(t+\Delta t) - f(t)$ by Δ***t***, moving to the limit in Δ***t*** →0 and taking into account (2.2) the following differential equation is ultimately derived:

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \lim_{\Delta t \to 0} \left[\alpha(t) \cdot \frac{y}{t} \right] = \alpha(t) \cdot \frac{y}{t} \tag{2.3}$$

which describes the rate of change of y productivity depending on the time elapsed after calving.

It is clearly evident that apart from the productive capacity of an animal ***y*** depends on the lactation period ***t*** as well which is shown in the equation (2.3) by the multiplier ***α(t)***. Here ***α(t)*** - is a decreasing and alternating function that defines the nature of the lactation curve and fulfills the following conditions (Smagin 2005):

$$1. \ \ \alpha'(t) < 0 \quad \text{and} \quad 2. \ \ \alpha(t) = \begin{cases} \geq 0 \ when \ \ t \in (0\,;\ t_{\max}] \\ < 0 \ when \ \ t \in (t_{\max};\, t_k \ ] \end{cases}. \tag{2.4}$$

Here $t_{\max}$ – - is the time in which the productive capacity reaches its peak $(\alpha(t) = 0)$ and t_k –is the time when lactation ends.

Dividing variables and integrating (2.3), we derive:

$$y(t) = e^{\int \frac{\alpha(t)}{t} dt} \tag{2.5}$$

This decision, depending on the type of function ***α(t)*** provides a range of models of lactation curves [15, 27-28, 36].

For example, if

$$\alpha(t) = b - 2ct,\ \alpha(t) = 1 - ct,\ \alpha(t) = b - ct \ ,$$

of the solution (2.5), respectively, we obtain models (1.4), (1.6) and (1.10).

Suppose that for the ***α (t)*** takes place:

$$\alpha(t) = -bt - 2ct^2 + \frac{d}{t} \ \text{and}\ \alpha(t) = b - ct + \frac{d}{t},$$

then we similarly have from (2.5), respectively Morant and Gnanasakthy (1989) and Keskin et al. (2010):

M16 $$y = \exp\left(a - bt - ct^2 - d/t\right) = e^{a-bt-ct^2-d/t} \quad (2.6)$$

M17 $$y = at^b e^{-ct-\frac{d}{t}} = at^b \exp\left(-ct - \frac{d}{t}\right) \quad (2.7)$$

Let $\alpha(t) = b - ct - 2dt^2 + \frac{q}{t} - kt \cdot \tan(kt)$. Considering it in (2.5), we obtain:

$$\int \frac{\alpha(t)}{t} dt = \int \left[\frac{b}{t} - c - 2dt + \frac{q}{t^2} - k\tan(kt)\right] dt =$$

$$= A + b \cdot \ln t - ct - dt^2 - \frac{q}{t} + \ln\left[\cos(kt)\right]$$

M18 $$y = at^b e^{-ct-dt^2-\frac{q}{t}} \cdot \cos(kt) \quad (2.8)$$

It should be noted that if the lactation curve equation is known then operational control of the milk production process can be achieved since with the dependence of $y(t)$ we can compute the maximum efficiency index and the corresponding time and also the amount of milk produced during each time interval.

3. Semi-empirical models

Among early attempts to apply mathematical methods to the study of biological problems one of the most successful outcomes occurred while studying the problem of growth of the total number of living organisms based on birth and death rates.

Verhulst (1838; 1845) and Pearl (1920) specified the mathematical model of logistic growth. The model examined the arresting factor of the population growth which lowered the growth rate dy/dt by a value proportional to the squared number. This more precise model results in the differential equation:

$$\frac{dy}{dt} \cdot \frac{1}{y} = r - \frac{r}{K} y \quad \text{or} \quad \frac{dy}{dt} = ry - \frac{r}{K} y^2 \quad (3.1)$$

Here it is assumed that ***y(t)*** describes the size of the population over time ***t***.

Where the parameter ***r*** is the specific rate of population growth; ***K*** is the maximum possible population size; t is time.

Equation (3.1) was derived under the following assumption: it is assumed that the dependence of the specific (average) growth rate of the population on its size is linear.

The Verhulst logistic equation is also referred to in the literature as the Verhulst-Pearl equation after Verhulst, who first derived the curve, and Pearl, who used the curve to approximate population growth in the United States in 1920.

The solution to this equation under the initial condition $y|_{t=0} = y_0$ is written as follows:

M19 $$y(t)=\frac{K}{1+(K/y_0-1)\cdot e^{-rt}} \quad \text{or} \quad y(t)=\frac{K}{1+\exp(a+bt)} \qquad (3.2)$$

The work (Kutsenko, 2012) states the mathematical description method for the time dependence of the total amount of milk yield from a cow in the lactation period $y(t)$ using Verhulst's logistic function obtained by solving the differential equation (3.1) over a limited period of growth in the size of biological populations as follows:

M20 $$y(t)=\frac{K}{1+\exp(a+bt)}+c+\frac{1}{2}qt^2 \qquad (3.3)$$

According to Lokhorst (1996) and Faridi et al. (2011) the over a limited period of feed intake equation as follows:

M21 $$y(t)=\frac{a}{1+b\exp(-act)}+dt+qt^2 \qquad (3.4)$$

Predictive calculations of the formulas (3.2) – (3.3) does not fully correspond to the true state at the initial and at the final stage of lactation curve. To eliminate these defects, generalizing the above models (3.2) - (3.4), we propose the following model

M22 $$y(t)=\frac{K}{1+\exp(a+bt)}+c+dt+f\,t^s \qquad (3.5)$$

where a, b, c, d, q, s – empirical parameters, which are based on experimental data on cow milk yield.

4. Materials and Methods

4.1. Database

In order to compare our proposed and previously used models (France, 1987, etc.), we identified the parameters of these models. Further, various statistical criteria was used to select the best model. For this purpose we used the data of experimental milk yields of cows (Keskin et. al, 2010) from a subsidiary farm of the Selcuk University (Konya, Turkey).

Milking was carried out in accordance with the recommendations of the Animal Science Department of the Agricultural Faculty.

Daily and weekly milk yields were used for each cow.

The observed total milk yield (TMY) per lactation was estimated using the Fleischmann's method (FLS) (Ruiz et al., 2000):

$$\mathbf{TMY}=(y_1\cdot t_1)+\sum_{i=2}^{k}\left[\frac{y_i+y_{i+1}}{2}(t_{i+1}-t_i)\right]$$

where TMY is total milk yield, y_1 is yield at first milk record, t_1 is interval between calving and first recording; y_i is yield of the record i and t_i is interval between the record i and the record (i +1), (i=1, . . . , k).

4.2. The Commonly used and proposed models

In the present work we have considered in the existing literature and our proposed models describing a lactation curve. The Commonly used models, which have been described above, is presented below (Tab 1).

Table 1. Existing empirical models used to describe the process of lactation

1*	$y = a + bt + ct^2$	8	$y = at^b e^{-ct-\frac{d}{t}}$
2	$y = a + bt + \frac{c}{t}$	9	$y = e^{a-bt-ct^2-\frac{d}{t}}$
3	$y = a + bt + ct^2 + dt^3$	10	$y = a + b\sqrt{t} + c\ln t$
4	$y = \frac{t}{a + bt + ct^2}$	11	$y = \frac{A}{1 + \exp(a + bt)}$
5	$y = a + ct + be^{-0{,}61t}$	12	$y = \frac{A}{1 + \exp(a + bt)} + c + \frac{1}{2}qt^2$
6	$y = a\left(1 - e^{-bt}\right) - c$	13	$y = a + bt + ct^2 + d\ln t + k(\ln t)^2$
7	$y = at^b e^{-ct}$	14	$y = a + bt + ct^2 + dt^3 + k\ln t + q(\ln t)^2 + r(\ln t)^3$

***1-**Dave, 1971; **2-** Bianchini, 1984; **3-** Dağ et al, 2005; **4-** Nelder, 1966; **5-** Wilmink, 1987; **6-**Cobby and Le Du, 1978; **7-** Wood, 1967; **8-** Keskin et al, 2010; **9-** Morant and Gnanasakthy, 1989; **10-** Guo and Swalve, 1995; **11-** Verhulst, 1938; **12-** Kutsenko, 2012; **13-** Ali and Schaffer, 1987; **14-** Mikayilov, 2014.

Generalizing the models (3.2) and (3.4) as well using the solution (2.5) of the differential equation (2.3), which describes the rate of productivity change, we obtained lactation models, that are given below (Tab 2).

Table 2. Suggested models used for the lactation process description

15	$y = a t^b e^{-ct-dt^2-\frac{q}{t}} \cdot \cos(kt)$
16	$y(t) = \frac{K}{1 + \exp(a + bt)} + c + dt + qt^s$

4.3. Statistical Analysis

For comparison of model fitting, we used the seven comparison criteria. These criteria are given below:

1. Root Mean Squared Error (RMSE) (also called the root mean square deviation, RMSD) is a frequently used measure of the difference between values predicted by a model and the values actually observed from the environment that is being modelled.

$$\sigma_{y/t} = \mathbf{RMSE} = \sqrt{\frac{1}{m}\sum_{i=1}^{n}\left(y_i - \tilde{y}_i\right)^2} \tag{4.1}$$

where, m=n-p-1 if n≤30 and m=n-p if to n>30; n ist the number of observations; p – ist the number of estimable parameters in the approximating linear and nonlinear empirical models, and should always be p<n; y_i: is actual milk yield values; $\tilde{y}_i$ is predicted milk yield values.

2. The coefficient of determination (R^2), which indicates the precision of the modelled daily milk yield in relation to the measured values.

$$R^2 = 1 - \sum_{i=1}^{n}\left(y_i - \tilde{y}_i\right)^2 / \sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2 \tag{4.2}$$

$\bar{y} = \sum y_i/n$ –is the average of the milk yield observed values. When R^2 is used as model selection criterion, we are faced with several problems [18, 38].

1) To compare the degree of success of different models using R^2, the functional structure of the model and estimators must be the same.

2) The R^2 value increases as the number of explanatory variables in the model increases.

However, this situation will lead to a rise in the variance of forecast errors. In this case, the use of the adjusted coefficient of determination is more appropriate. Therefore, it is more appropriate to use the adjusted coefficient of determination in nonlinear models [34].

3. Adjusted coefficient of determination. The use of an adjusted R^2 (often written as R_{adj}^2) is an attempt to take account of the phenomenon of the R^2 automatically and spuriously increasing when extra explanatory variables are added to the model. The adjusted R^2 is defined as:

$$R_{adj}^2 = 1 - \left(1 - R^2\right)\frac{n-1}{n-p} \tag{4.3}$$

4. The agreement index (D), which indicates the accuracy of the modelled daily milk yield in relation to the measured values [43], and is given by:

$$\mathbf{D} = 1 - \frac{\sum_{i=1}^{n}\left(y_i - \tilde{y}_i\right)^2}{\sum_{i=1}^{n}\left[\left|y_i - \bar{y}\right| + \left|\tilde{y}_i - \bar{y}\right|\right]^2} \tag{4.4}$$

The index **D** ranges from 0 to 1, where the value 1 means a perfect accuracy of the estimated data, and the value 0 means that there is no accuracy.

5. For the purposes of determining the adequacy of the models used Theil (1966) coefficient mismatch, which is calculated according to the equation:

$$\mathbf{U1} = \frac{\sqrt{\sum_{i=1}^{n}\left(y_i - \tilde{y}_i\right)^2}}{\sqrt{\sum_{i=1}^{n} y_i^2} + \sqrt{\sum_{i=1}^{n} \tilde{y}_i^2}} \qquad (4.5)$$

Theil's inequality coefficient is useful for comparing different forecast methods. To interpret the U statistics the general guide is: **U1** is bound between 0 and 1, with values closer to 0 indicating greater forecasting accuracy.

6. The Mean Absolute Percentage Error (MAPE), also known as Mean Absolute Percentage Deviation (MAPD), is a measure of prediction accuracy of in statistics and is the most common measure of forecast error. Is usually expresses accuracy as a percentage, and is defined by the formula:

$$\mathrm{A} = \mathbf{MAPE} = \frac{100\%}{n}\sum_{i=1}^{n}\left|\frac{y_i - \tilde{y}_i}{y_i}\right| \qquad (4.6)$$

7. Akaike information criterion (AIC). The Akaike information criterion (AIC) [1] is a popular method for comparing the adequacy of simple and multiple linear or nonlinear and nested or non-nested models. It is often used to select the model [7], which is calculated according to the equations:

$$\mathbf{AIC} = \begin{cases} \ln\left[\frac{1}{n}\sum_{i=1}^{n}\left(y_i - \tilde{y}_i\right)^2\right] + \frac{2p}{n}, & if \quad \frac{n}{p} \geq 40 \\ \ln\left[\frac{1}{n}\sum_{i=1}^{n}\left(y_i - \tilde{y}_i\right)^2\right] + \frac{2p}{n-\left(p+1\right)}, & if \quad \frac{n}{p} < 40 \end{cases} \qquad (4.7)$$

5. Results and Discussion

For comparison the existing in literature (Tab 1) and our proposed models (Tab 2), carried out the identification of parameters as well calculated statistical criteria for comparison these models.

For this purpose the parameters of existing and suggested patterns based on data of yields from two cows from a part-time farm of the University Zootechnics Chair “Selchuk” (Konya, Turkey) have been defined.

For computation of statistical parameters was used the program pack Statıstıka 5.0 using the Quasi-Newton and Rosenbrock and Quasi-Newton methods. The calculation results are shown in Table3.

Table 3. Comparison of common and proposed models to describe the process of lactation cows

Models	Statistical parameters values for the existing models													
	RMSE		**R^2, %**		**R^2_{adj}, %**		**D**		**UI**		**A, %**		**AIC**	
	y^*_1	**y_2**	**y_1**	**y_2**	**y_1**	**y_2**	**y_1**	**y_2**	**y_1**	**y_2**	**y_1**	**y_2**	**y_1**	**y_2**
1	1,73	1,56	62,54	73,32	87,59	91,87	0,876	0,919	0,049	0,046	8,445	7,295	1,112	0,904
2	1,74	1,56	62,19	73,43	87,34	91,90	0,873	0,919	0,049	0,046	8,325	7,862	1,121	0,899
3	1,72	1,56	63,28	73,35	88,03	91,92	0,880	0,919	0,049	0,046	8,185	7,197	1,096	0,906
4	1,82	1,70	58,61	68,33	85,27	89,58	0,853	0,896	0,052	0,050	8,573	8,994	1,212	1,075
5	1,77	1,58	61,14	72,76	86,88	91,65	0,869	0,917	0,050	0,047	8,343	8,001	1,149	0,924
6	1,70	1,56	64,09	73,37	88,23	91,89	0,882	0,919	0,048	0,046	8,207	7,179	1,074	0,901
7	1,77	1,63	60,92	71,05	86,57	90,86	0,866	0,909	0,050	0,048	8,602	8,218	1,154	0,985
8	1,76	1,61	61,20	71,60	86,78	91,11	0,868	0,911	0,050	0,047	8,512	8,244	1,151	0,969
9	1,70	1,51	63,80	75,09	88,03	92,46	0,880	0,925	0,048	0,044	8,407	7,414	1,081	0,838
10	1,77	1,62	60,83	71,20	86,68	91,04	0,867	0,910	0,050	0,048	8,558	8,274	1,157	0,980
11	1,73	1,56	62,98	73,61	87,63	91,95	0,876	0,920	0,049	0,046	8,382	7,202	1,104	0,896
12	1,58	1,44	69,24	77,36	90,33	93,34	0,903	0,933	0,045	0,042	7,315	6,850	0,930	0,746
13	1,65	1,44	66,18	77,39	89,25	93,34	0,893	0,933	0,047	0,042	7,991	6,811	1,017	0,745
14	1,55	1,43	69,90	77,64	90,74	93,49	0,907	0,935	0,044	0,042	7,280	6,781	0,908	0,740
Models	Statistical parameters values for the proposed models													
	RMSE		**R^2, %**		**R^2_{adj}, %**		**D**		**UI**		**A, %**		**AIC**	
	y_1	**y_2**	**y_1**	**y_2**	**y_1**	**y_2**	**y_1**	**y_2**	**y_1**	**y_2**	**y_1**	**y_2**	**y_1**	**y_2**
15	1,56	1,43	69,85	77,76	90,73	93,49	0,907	0,935	0,044	0,042	7,201	6,776	0,906	0,731
16	1,34	1,40	77,94	78,77	93,63	93,80	0,936	0,938	0,038	0,041	5,664	6,473	0,609	0,698

* Duration of lactation for cows $y_1(t)$ 259 days, and for $y_2(t)$ 308 days.

To select the best model, the values of the statistical parameters calculated utilizing equations (4.1) - (4.7) are given in Table 3.

The existing models in the literature and our proposed models (Table 2 and 3) used to describe the lactation process are compared with each other using these values, and the best model is selected.

According to these results (Table 3), it is clear that the best model for both cows among the existing ones is Model 14 proposed by Mikayilov (2013) and among the proposed is the Model 16, when all the seven criteria are taken into account.

When statistical results related to the measured values and values calculated according to all the above models are examined, it is seen that the maximum (R^2=78.20% and 78.97%, R^2_{adj}=77.94% and 78.77%, D=0.9363 and 0.9383) and minimum (RMSE=1.3383 and 1.4017, UI=0.0376 and 0.0410, A=5.66% and 6.47, AIC=0.6094 and 0.6978) values for both cows are obtained for proposed model 16:

$$y(t) = \frac{K}{1+\exp(a+bt)} + c + dt + qt^s \quad (5.1)$$

Once a model was chosen as the best, total milk yield per lactation (ETMYL) was estimated for the selected equation (5.1). Using the expression (5.1) and following the formula:

$$y_{total} = \int_{t_0}^{t_f} y(t)\,dt = \left\{\frac{K}{b}\left[(a+bt) - \ln\left(1+e^{a+bt}\right)\right]\right\}_{t_0}^{t_f} + \left(ct + \frac{1}{2}dt^2 + \frac{q}{s+1}t^{s+1}\right)_{t_0}^{t_f} \quad (5.2)$$

we derive the milk output for a given animal in a period of time between moments t_{i-1} and t_i, and also the total yield time between moments t_0 and t_f.

One of the main characteristics, i.e. milk yield to day 259 (for y_1) and day 308 (for y_2) of the overall lactation curve were estimated using the proposed model (5.1).

The results of calculations for these models were compared with the observed values, which are as follows:

t	Total yield, y_{total}		**A**
Day	y_{obs}	y_{16}	**%**
259	4491,30	4482,8	0,19
308	5132,20	5123,7	0,17

The results of calculations, calculated using the formula (5.2), very close to the observed values

Thus, using the values of the model's parameters (5.1) and the relation (5.2) we can evaluate the yield at an arbitrary lactation point. Such a procedure can be useful in modeling the processes involved in milk production, for example, calculating feed rations.

Conclusion

It should be noted that earlier, when choosing a models, preference was given to models with fewer parameters, in order to simplify the calculation. However, given the advances in computer technology in today's world, the number of parameters is no longer relevant as a criterion. It is certain that, despite a heavy computational burden, the model that is thought to be closest to reality according to statistical analyses should be preferred.

Given the results obtained, it would be wrong to be content with the 2 or 3 simple classical criteria, which yield misleading results in the selection of the most appropriate model in pursuit of solutions to such problems.

It is thought that the application of detailed analyses is more appropriate in model selection.

Comparison of statistical parameters revealed that among all studied models the proposeds model 16 best describes the yields' data for both cows. The models presented in this paper require further theoretical and experimental development and testing under varying conditions and on different animals. The aims of further investigations into the problems of tracing dynamic models for the elaboration of animals' productivity and methods for tracing lactation curves must include the following:

- consideration of the most important properties of the physiological state of animals in the lactation period such as initial, maximal and final daily milk yields;
- calculation of statistical parameters according to the above-mentioned models based on experimental data;
- selection of the best-fit lactation curve models.

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